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# On the work of J. ECALLE(Algebraic Analysis)

AUTHOR(S):

Malgrange, B.

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# On the work of J. ECALLE

B. Malgrange

I. Let  $D$  be a small disc  $\{|z| < \epsilon\}$  in  $\mathbb{C}$ ; write  $D^* = D - \{0\}$ ,  $\tilde{D}^*$  = the universal covering of  $D^*$  with some fixed base-point  $a \in D^*$ .

Put  $\tilde{\mathcal{O}} = \mathcal{O}(\tilde{D}^*)$ , the space of holomorphic functions on  $\tilde{D}^*$ , and  $\tilde{\mathcal{E}} = \tilde{\mathcal{O}}/\mathcal{O}(D)$ ; one has two well-known morphisms  $\tilde{\mathcal{O}} \xrightleftharpoons[\text{var}]{\text{can}} \tilde{\mathcal{E}}$ , where "can" is the quotient map, and "var" is defined by  $\text{var} \cdot \text{can} = I \cdot \text{id}$ ; here  $I$  is the action of the monodromy on  $\tilde{\mathcal{O}}$ . The space  $\tilde{\mathcal{E}}$  can be considered as a space of microfunctions at  $0 \in \mathbb{R}$ ; on  $\tilde{\mathcal{E}}$ , the convolution product  $(f, g) \mapsto f * g$  is well defined, with all the usual properties.

Now, let  $\Omega$  be a discrete subgroup of  $\mathbb{C}$ , for instance  $\Omega = \mathbb{Z}$ ; we suppose  $D \cap \Omega = \{0\}$ .

Definition 1 We denote by  $\mathcal{C}(\Omega)$  the set of the  $f \in \tilde{\mathcal{E}}$  such that  $\text{var } f$  has an analytic continuation to the whole space  $\mathbb{C} - \Omega$ ; here,  $\mathbb{C} - \Omega$  denotes the universal covering of  $\mathbb{C} - \Omega$  with the same base-point  $a \in \tilde{D}^*$  as before.

Theorem 2 (Ecalé).  $\mathcal{C}(\Omega)$  is a convolution subalgebra of  $\tilde{\mathcal{E}}$ .

This convolution algebra is the basic object of Ecalé's theory. A important result is the description of the singularities of  $f * g$  ( $f, g \in \mathcal{C}(\Omega)$ ) in terms of the singularities of  $f$  and  $g$ ; this is done with the introduction of "alien derivations"; for more precise statements, see (E1).

II. Let  $G$  be the group of germs of analytic automorphisms  $(\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$ , and

let  $H$  be the subgroup of germs tangent to identity. One of the works of Ecalé is the classification of conjugacy classes in  $H$ ; two methods are given to describe that classification; for the first one, I refer to [E2] or [M]; I will describe briefly the second one.

For  $h \in \mathbb{C}[[z]]$ , write  $h(z) = z + a_2 z^2 + \dots$ ; using the fact that  $h$  can be formally embedded in a one-parameter group, it is easy to prove that the formal conjugacy class of  $h$  is determined by two invariants

- i) The pair  $(n, a_n)$ , where  $n = \inf \{m \mid a_m \neq 0\}$
- ii) The coefficient of  $\frac{1}{z}$  in  $\frac{1}{h(z)-z}$  (this depends only on  $a_n, \dots, a_{2n-1}$ )

Then one has to find the analytic invariants corresponding to a given formal class. For simplicity, I will consider only the formal class defined by  $a_2 = a_3 = 1$ . Then we have

$h(z) = z + z^2 + z^3 + a_4 z^4 + \dots$ ; write  $h_0(z) = \frac{z}{1-z}$ . After the change of variable  $z = \frac{1}{\xi}$ ,  $g = \frac{1}{h}$ , one has  $g(\xi) = \xi - 1 + a_2 \xi^{-2} + a_3 \xi^{-3} + \dots$ , and  $g_0(\xi) = \xi - 1$ ; one sees easily that there exist one and only one fixed power series

$\varphi(\xi) = \xi - c_1 \xi^{-1} + c_2 \xi^{-2} + \dots$ , which satisfies  $\varphi \circ g = g \circ \varphi$ .

Let  $\tilde{\Phi}$  be the Fourier-Borel transform of  $\varphi$ , i.e. the formal microfunction

$$\tilde{\Phi} = \delta' + \sum_{k \geq 0} c_k \frac{z^{k-1}}{(k-1)!} \gamma \quad (\gamma = \text{the Heaviside function})$$

one sees in fact  $\tilde{\Phi} \in \tilde{\mathcal{E}}$ ; a much stronger result is the following

Theorem 3 (Ecahle, [E2]) If  $\Omega$  denotes the set  $2\pi i \mathbb{Z}$ , one has:  $\tilde{\Phi} \in \mathcal{C}(\Omega)$

Ecahle prove more precise results; consider for instance the analytic continuation of  $\tilde{\Phi}$  at the half-plane  $\operatorname{Re} x > 0$ , and denote by  $\tilde{\Phi}_\eta$  the microfunction that this continuation determines at the point  $2\pi i \eta$ ; then one has, for some  $c_n \in \mathbb{C}$  and  $p_n$  holomorphic near  $2\pi i \eta$ :  $\tilde{\Phi}_\eta = c_n \delta(x - 2\pi i \eta) + p_n \gamma(x - 2\pi i \eta)$ . Ecahle give some functional equation for  $\tilde{\Phi}_\eta$  (and other singularities over  $2\pi i \mathbb{Z}$  of  $\tilde{\Phi}$ ), in terms of other derivations, and proves also that the  $c_n$  give a complete list of analytic invariants.

#### References

- [E1], [E2] J. Ecahle, Théorie des fonctions résurgentes, ~~2~~ vol 1 and 2, Publications Mathématiques de l'Université d'Orsay, (1981-82)
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